

# Rényi statistics in high energy particle production

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## Abstract

It is shown that Rényi statistics provides a plausible basis to describe the hadron distributions measured in high energy particle interactions. Generalized Boltzmann and gamma distributions obtained by maximization of Rényi entropy under constraints on Kolmogorov-Nagumo averages are used to describe the hadron transverse momentum and multiplicity spectra correspondingly.

A multi-hadron production in high energy hadronic collisions is a complicated phenomenon generally modeled by a dynamic system of hadronizing quarks and gluons with many degrees of freedom and high level of correlations. Though using QCD one calculates elementary interactions of quarks and gluons at microscopic level, a complexity of the system generally doesn't allow an extension of these calculations to predict the observed particle spectra. Thus, it is tempting to apply statistical methods to understand properties of particle production presented by the experimentally measured statistical distributions. In this paper we consider two distributions most systematically studied in experiments with  $pp$  and  $\bar{p}p$  interactions: the inclusive charged particle transverse momentum spectrum and charged particle multiplicity distribution.

At low collision energy ( $s$ ) an exponential behavior of the inclusive single particle spectra as function of particle's transverse momentum ( $P_t$ ) has been

observed. These spectra are interpreted using a thermodynamic analogy and described by a Boltzmann-type distribution [1] <sup>1</sup>

$$\left. \frac{d^3\sigma}{d^3p} \right|_{y=0} = A \cdot e^{-E_t/T}, \quad (1)$$

where  $A$  is a normalization factor,  $E_t = \sqrt{m^2 + P_t^2}$  is a transverse energy of the produced particle with mass  $m$ , and  $T$  is a characteristic temperature of the hadronizing system. What is a genesis of the exponential distribution?

The least biased method to obtain statistical distributions, which are realized in the nature was promoted by E.T.Jaynes as Maximum Entropy Principle (MEP) [9]. The MEP states that the physical observable has a distribution, consistent with given constraints which maximizes the entropy. The Boltzmann exponential distribution (1) arises naturally from a maximization of Gibbs-Shannon entropy

$$S_{GS} = - \sum_n P_n \ln(P_n) \quad (2)$$

under a constraint on an average energy of produced hadrons.

The data on high- $s$  interactions show a deviation from the exponential towards a power-law behaviour and can be approximated [2] by

$$\left. \frac{d^3\sigma}{d^3p} \right|_{y=0} = A \cdot \left(1 + \frac{E_t}{\kappa T}\right)^{-\kappa}. \quad (3)$$

At finite values of the parameter  $\kappa$  (3) represents a power-law distribution, while in the limit of  $\kappa \rightarrow \infty$  the expression (3) reduces to Boltzmann exponent (1). In plasma physics the  $\kappa$ -distribution (3) is frequently used to describe an excess of highly energetic particles with respect to that expected from an exponential spectrum [3]. The enhanced probability of the high energy particle fluctuations (or an appearance of a *heavy tail* in a distribution) could be viewed as a collective phenomenon resulting from a strong intrinsic correlation built in the system. The heavy-tailed distribution (3) arises in a framework of non-extensive statistical mechanics [4] as a result of maximization of Harvda-Charvat-Daróczy-Tsallis (HCDT) entropy [5]

$$S_q = \frac{1 - \sum P_n^q}{q - 1}. \quad (4)$$

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<sup>1</sup>Since the most data on hadron production are available for central rapidities ( $y = 0$ ) we consider the differential cross sections for central rapidity only.

under a constraint on an average energy of the particles. Formally, the expression (3) with  $\kappa = (q-1)^{-1}$  is a non-extensive generalization of Boltzmann distribution.

It is interesting to note that the exponential distribution (1) transforms in to (3) when a value of parameter  $T$  fluctuates according to a gamma distribution [6]

$$f(T, \lambda, k) = \frac{\lambda(\lambda T)^\mu \cdot e^{-\lambda T}}{\Gamma(k)} . \quad (5)$$

The gamma distribution represents a sum of  $\mu + 1$  independent exponentially distributed random variables and arises naturally from maximization of Gibbs-Shannon entropy under constraints on the mean values of  $T$  and  $\ln(T)$ . The latter requirement corresponds to a constraint on a geometric mean.

There are a number of similarities between a behaviour of hadron  $P_t$  spectra and hadron multiplicity distributions. The multiplicity distributions are observed to be significantly broader (*over-dispersed*) than the Poisson distribution expected for uncorrelated systems. The data on charged hadron multiplicities are frequently parameterized by the Negative Binomial Distribution (NBD) [7]. The NBD arises from Poisson distribution with the mean value fluctuating according to a gamma distribution. In other words, the NBD results from a discrete gamma distribution with each point smeared by a correspondent Poisson distribution. Like the heavy tail in hadron  $P_t$  spectra, an over-dispersion of hadron multiplicity distribution is believed to be a result of strong long-range correlations in the system. It is therefore tempting to find a common statistical approach in which both the heavy-tailed and over-dispersed distributions arise naturally. Below it is shown that Rényi statistics with Kolmogorov-Nagumo averages provides a plausible basis to derive such statistical distributions.

The generalized Kolmogorov-Nagumo (KN) average of a variable  $x_n$  is defined as

$$\langle x \rangle = \phi^{-1} \left( \sum_n P_n \phi(x_n) \right) , \quad (6)$$

where  $\phi$  is a monotonic function and  $P_n$  is a probability of  $n$ -th state. For  $x_n = \ln(P_n)$  the average (6) represents a generalized measure of information (entropy). If one requires the generalized entropy to be additive (extensive) and  $\langle x + c \rangle = \langle x \rangle + c$ , where  $c$  is a constant, only linear and exponential

functions  $\phi(x)$  satisfy these requirements. It is practical to represent the function  $\phi(x)$  in a generalized form

$$\phi(x) = \frac{e^{(1-q)x} - 1}{1 - q}, \quad (7)$$

which reduces to a linear function in the limit  $q \rightarrow 1$ . The linear function leads to the traditional definition of an average and to Gibbs-Shannon entropy (2), while the generalized function (7) leads to a definition of Rényi entropy [8]

$$S_q = \frac{\ln(\sum P_n^q)}{1 - q}. \quad (8)$$

For  $|q - 1| \ll 1$  the  $S_q$  can be approximated by non-extensive HCDT entropy (4), while for  $q \rightarrow 1$  the entropy (8) reduces to Gibbs-Shannon form (2). The Gibbs-Shannon statistics has no built-in intrinsic correlations, such that for two variables  $x$  and  $y$  holds  $\langle x + y \rangle = \langle x \rangle + \langle y \rangle$ , which is not necessarily a case for the Rényi statistics. Although Rényi entropy provides the most general form of information measure, it has found so far a limited number of applications. One of the reasons for that is a little understanding of physical meaning of the entropic index  $q$ . Since for  $q = 1$  the Gibbs-Shannon statistics is restored it is reasonable to assume that a deviation of the entropic index value from unit is related to intrinsic correlations built up in a system. As we will see below, this intuitive conjecture is compatible with the hadronic interaction data.

In order to obtain a generalized Boltzmann distribution arising in the Rényi statistics one has to maximize  $S_q$  under a constraint on a KN-average value. This was done in [10] and in application to a hadron  $P_t$  spectrum can be rewritten as

$$\left. \frac{d^3\sigma}{d^3p} \right|_{y=0} = A \cdot (1 - \beta + \beta \cdot e^{\frac{(q-1)\lambda E_t}{\beta}})^{-\frac{1}{q-1}}, \quad q > 1 \quad (9)$$

with a parameter  $\lambda = (qT)^{-1}$  needed to make the energy dimensionless and  $\beta$  being Lagrange multiplier. For  $|(q-1)\lambda E_t/\beta| \ll 1$  the expression (9) is approximated by the power-law distribution (3) and the Boltzmann exponential distribution (1) follows in the limit  $q \rightarrow 1$ .

In Fig. 1 the experimental spectra measured at different  $pp$  and  $\bar{p}p$  collision energies are shown together with the parameterization (9). The experimental data come from [11]. In order to make a comparison with measured

differential charged particle cross section  $Ed^3\sigma/d^3p \equiv d^2\sigma/(2\pi P_t dP_t dy)$  at  $y = 0$  the right part of (9) is multiplied by  $E_t$  and  $m$  in  $E_t$  fixed to charged pion mass value. For all data sets a good data description in Fig. 1 is observed. To compare a quality of the data description using the power-law (3) and the generalized Boltzmann (9) distributions the residuals of the correspondent fits to the data are shown in Fig. 2 for two highest available collision energies. It is clear that the power-law function has local systematic deficiencies, while the generalized Boltzmann distribution describes the data equally well in the whole available range of the transverse momenta.

In order to derive a generalized gamma distribution one has to maximize entropy  $S_q$  under constraints on KN-mean values of  $x_n$  and  $\ln(x_n)$

$$\frac{\ln(\sum P_n e^{\frac{\lambda x_n(1-q)}{\beta}} - 1)}{1-q} \equiv \langle x \rangle, \quad \frac{\ln((\sum P_n x_n^{\mu(1-q)}) - 1)}{1-q} \equiv \langle \ln(x) \rangle. \quad (10)$$

In our application,  $x_n = N$  is the number of produced charged hadrons and the resulting distribution is read

$$P_N = A \cdot ((\lambda N)^{-\mu(q-1)} - \beta + \beta e^{\frac{(q-1)\lambda N}{\beta}})^{-\frac{1}{q-1}}, \quad q > 1, \quad (11)$$

For  $q \rightarrow 1$  the (11) is reduced to the gamma distribution (5).

In Fig. 3 the experimental charged multiplicity spectra measured at different collision energies [12] are fit to the parameterization (11). For all data sets a good description by the generalized gamma distribution (11) is found. The fits to the data show that the parameter  $q$  increases logarithmically with  $s$  both for  $P_t$  and multiplicity spectra. At the same time the experiments show a similar logarithmic rise of a particle-particle correlation strength with  $s$  [13]. This observation provides a support for the conjecture that the entropic index  $q$  is related to the intrinsic correlations in system. Note, that for  $\mu = 0$  the expression (11) transforms in to the generalized Boltzmann exponential distribution (9).

In conclusion, using the Maximum Entropy Principle two generalized statistical distributions have been derived from a maximization of Rényi entropy. These distributions have been applied to describe the experimental data on particle production in high energy  $pp$  and  $\bar{p}p$  interactions. The generalized Boltzmann distribution turned out to describe the  $P_t$  spectra of produced charged particles better than the traditionally used power-law distribution. The generalized gamma distribution provides a good description of the charged particle multiplicity.

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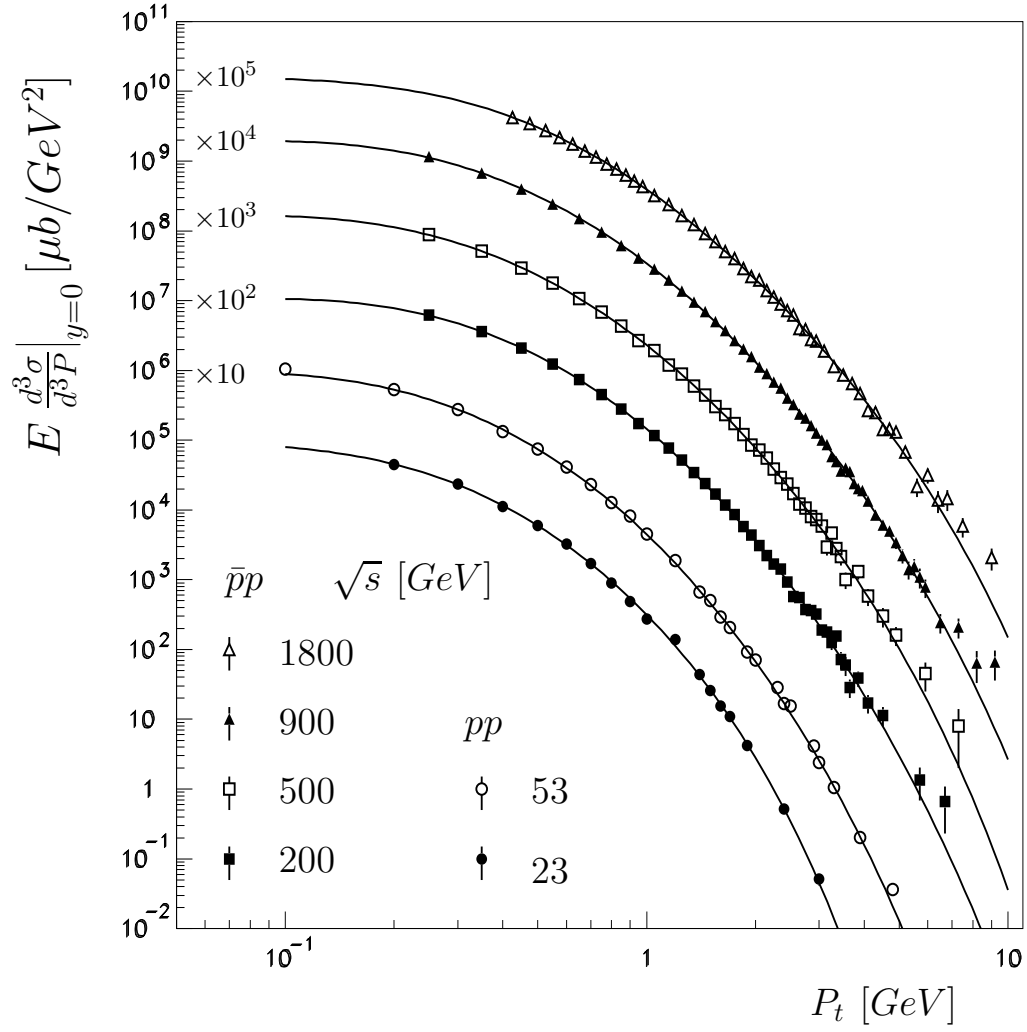


Figure 1: Measured charged particle  $P_t$  spectra overlayed with fits to the generalized Boltzmann distribution.

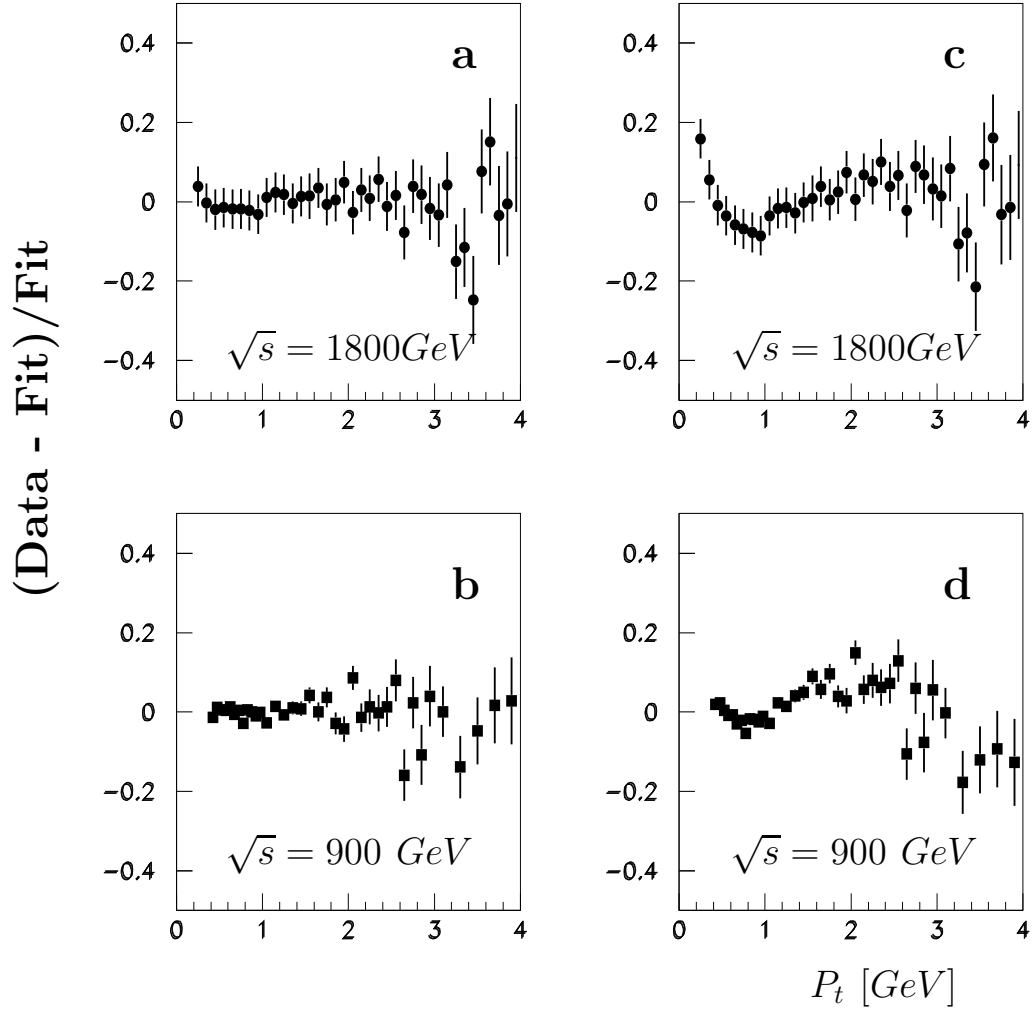


Figure 2: A relative difference between the measured charged particle  $P_t$  spectra and the fit results using the generalized Boltzmann (a and b) and the power-law HCDT parameterization (c and d).

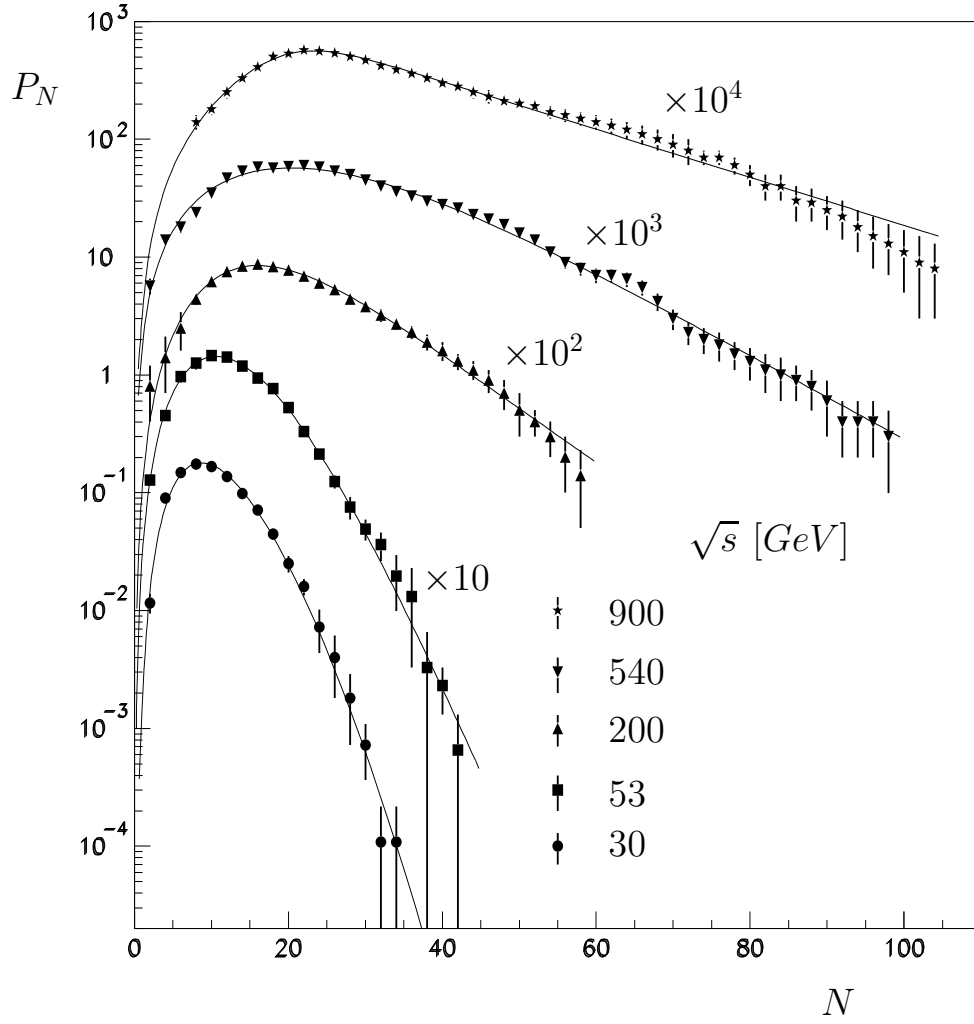


Figure 3: Measured charged particle multiplicity spectra overlayed with fits to the generalized gamma distribution.